How to Guard an Art Gallery

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May 2018

Introduction

In 1973, Victor Klee from the University of Washington posed the following problem: *"Consider an art gallery whose floor plan can be modeled by a polygon with n vertices. How many stationary guards would it take to guard the entire floor plan?"* Klee, recently the president of the Mathematical Association of America, had considerable outreach, and within a few years work had been accomplished to form conclusions to his initial question and questions that followed. This paper will delve into a number of these questions and consider different scenarios in which the results can be applied.

Initial Question

Klee's initial question has a couple of different interpretations, as one can consider the number of guards needed based on the number of vertices or the number of walls. For the purpose of this paper, we will be trying to determine for some number of walls. Regarding notation, this paper will use the following:

Let

G be some arbitrary art gallery,

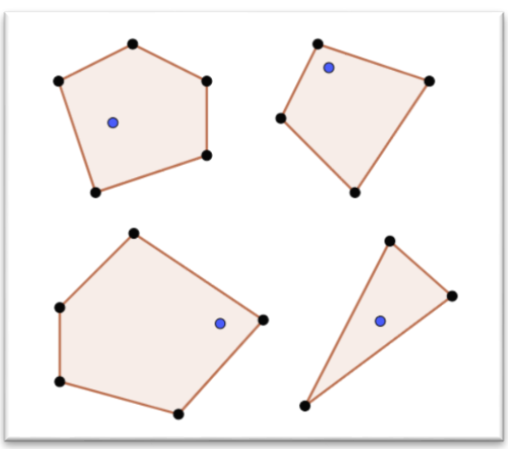
w be the number of walls in a gallery,

G_w be some art gallery G with w walls,

$guard(G)$ = the *minimum* number of guards needed to protect the art gallery G.

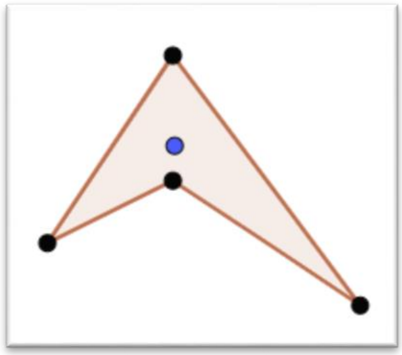
Klee's initial question was to determine $guard(G)$, the minimum number of guards required to guard some specific gallery G. In order to establish that $guard(G) = g$, we determine two conditions to be true. First, gallery G can truly be guarded by g guards. This shows $guard(G) \leq g$. Second, G cannot be guarded by fewer than g guards. That is, $guard(G) \geq g$.

Initially, there are trivial results that provide little insight to the solution to a complex problem. We first note that any *convex* polygon can be guarded by a single guard.

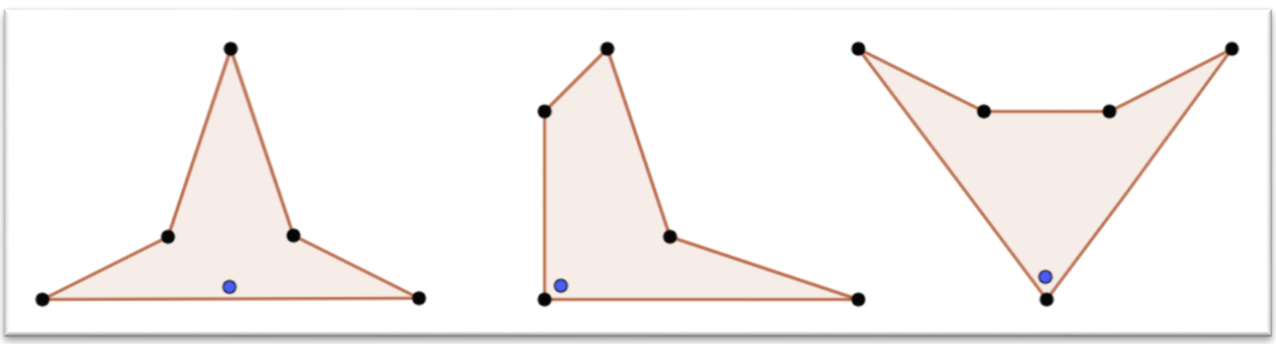


For any convex polygon (gallery), a guard can be placed anywhere on the interior of the polygon and have sight line to the entire floor area of the gallery. Similarly, polygons with a small number of sides yield little insight to the complexity and solution to this problem. All triangles are convex, so it a natural understanding that all triangles with three walls can be covered by a single guard. While there is more to consider, it can also be shown that all 4-walled and 5-

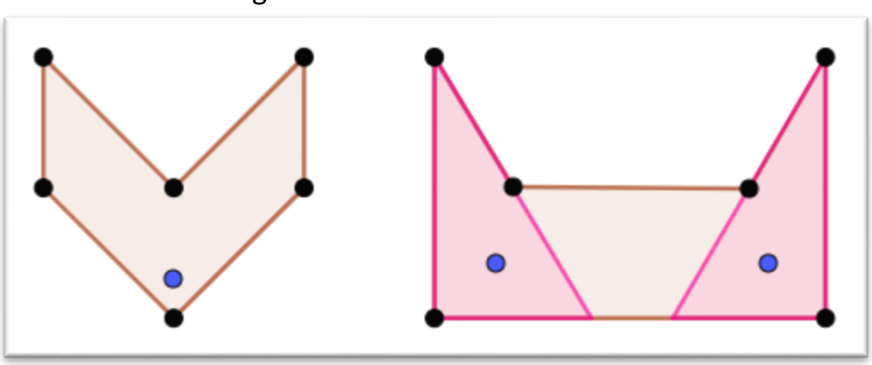
walled galleries can be guarded with a single guard. When considering 4-walled galleries, we are limited to two basic configurations. The first is convex, and the second consists if variations where a single vertex is greater than 180 degrees. Note, it is impossible for more than one vertex of a quadrilateral to be greater than 180 degrees due to angle sum rules. Thus, we have the resulting single basic configuration.



It can be shown that any 4-walled gallery can be guarded by a single guard. Specifically, this can be true by placing a guard at the vertex that is on the interior of the line connecting some two vertex points. A similar argument can be made for 5-walled polygons as there three different concave configurations with one or two vertices measuring greater than 180 degrees. In all scenarios fairly simple arguments can be made to show a single guard can guard the interior of the polygon.



Six-walled galleries are the first that require one or two guards depending on configuration, as shown in the diagram below.



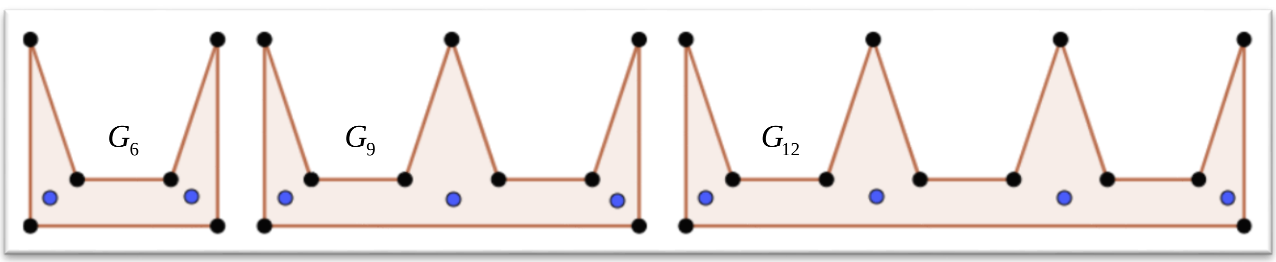
The figure on the right requires two guards because a guard must be placed somewhere in each of the pink triangular regions to be able to see the upper vertices. However, as the number of walls/vertices increases, the complexity of being able to determine a value for $guard(G)$ increases greatly. While it is fairly easy to determine a way in which a gallery can be guarded, it is significantly more difficult to determine if that number of guards is minimal. As a result, there is currently no efficient algorithm for determining a placement of guards to calculate $guard(G)$, and researchers in computational complexity believe there never will be. Thus, Klee's initial question yields no results.

Klee's Question Reconsidered

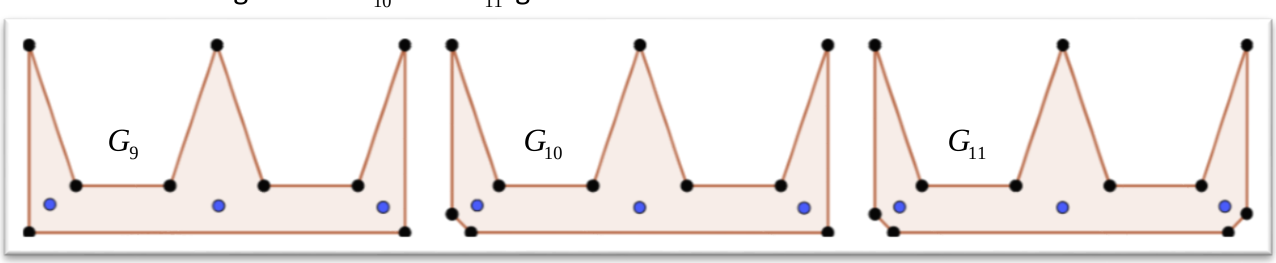
While Klee's initial question remains unanswered, there are many viable related questions that have solutions. The first prominent solution answers the question, ""If I am presented with the task of protecting ANY gallery with w walls, but no idea of the floor plan, what is the minimum number of guards I would have to send that is GUARANTEED to protect the gallery?" Let $g(w) =$ the maximum number of guards required for all art galleries with w walls, regardless of configuration. In order to establish $g(w) = g$, it is once again necessary to meet two conditions.

1) Every gallery can be protected by g guards ($g(w) \leq g$).
2) There exists some gallery G that cannot be protected by fewer than g guards ($g(w) \geq g$).

As we saw previously, the first condition can be met simply by providing some singular example in which a gallery would require a certain number of guards. Consider then the *crown galleries*. Crown galleries are named on their basic shape and are useful galleries in trying to determine an upper bound for $g(w)$. Some sample crown galleries are shown below.



In each instance of the crown gallery, every "point" of the crown requires a guard, so it would appear that for any instance of G_w crown gallery where $w = 3k$, we would require $w/3$ guards. Also consider the galleries G_{10} and G_{11} galleries shown below.



Each of the galleries represents the same basic configuration. The addition of the new sides in G_{10} and G_{11} can be added for any basic crown G_{3k}. Thus, we have $guard(G_{3k}) = guard(G_{3k+1}) = guard(G_{3k+2})$ for crown galleries which implies $g(3k) = g(3k+1) = g(3k+2)$. Since this is true, we have that $g(w) \geq \lfloor w/3 \rfloor$. At this point we do not know that crown galleries require the most guards for some w - walled gallery, but we have established a *potential* value for the first condition of calculating $g(w)$.

In 1975, however, it was confirmed that the crown galleries actually do represent the scenario requiring the largest number of guards. Vašek Chvátal of Stanford University was able to establish that any gallery with w walls was indeed able to be guarded by at most Chvátal guards. When combined with the previous result, we have $g(w) \geq \lfloor w/3 \rfloor$ and $g(w) \leq \lfloor w/3 \rfloor$. Thus, we have the

Art Gallery Theorem.

We have $g(w) = \left\lfloor \dfrac{w}{3} \right\rfloor$ for $w = 3, 4, 5...$

In other words, $\left\lfloor \dfrac{w}{3} \right\rfloor$ walls are sufficient, and sometimes necessary, to protect a gallery with w walls.

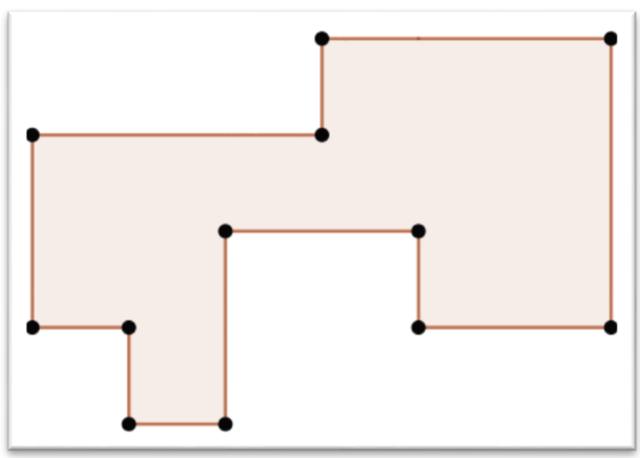
Chvátal's argument used induction on the number of walls. While this was widely accepted, the proof was considered difficult to follow.

Chvátal's proof was explained in more simplistic terms by mathematician Steve Fisk of Bowdoin College in 1977. It was so clear, in fact, that his proof was included in *"Proofs from THE BOOK",* a collection of the most elegant proofs of theorems. Fisk's argument operates under two different (proven) claims:

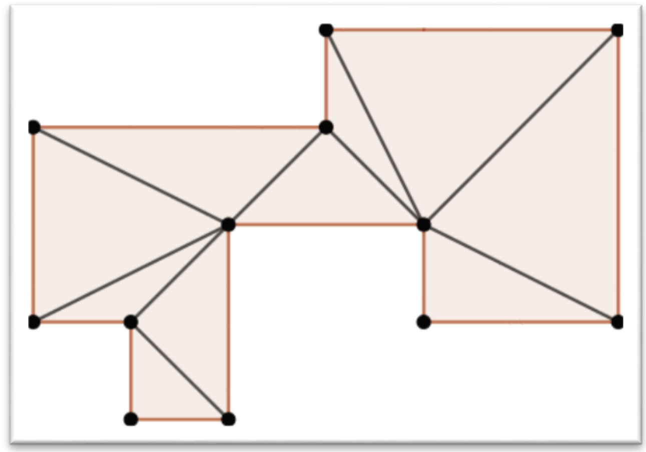
1) Every art gallery has a triangulation by diagonals.
2) Every such triangulation has a polychromatic three coloring.

Fisk's proof is as follows:

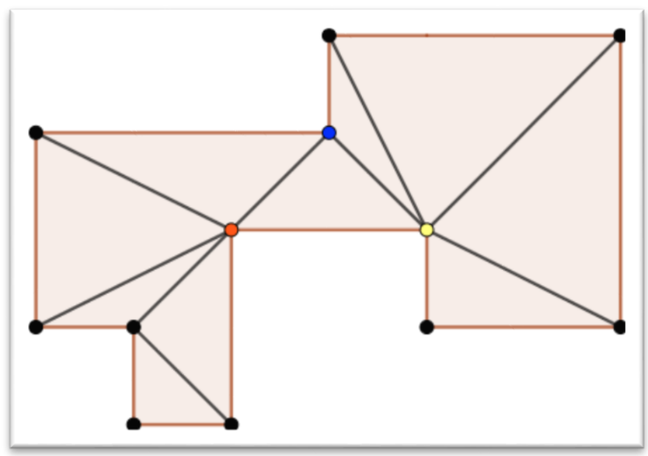
1) Consider any arbitrary polygon with w walls.



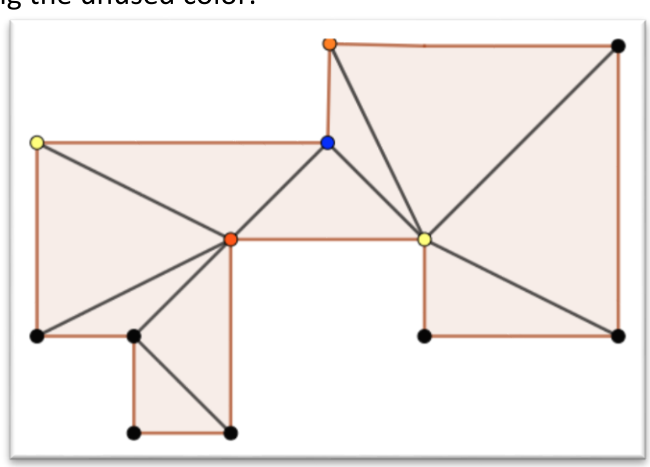
2) Construct a triangulation by diagonals (all polygons have a diagonal triangulation – claim 1).



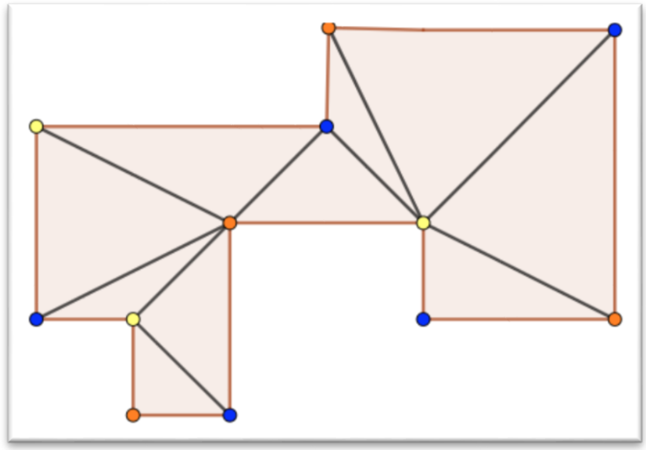
3) Pick any triangle and color the vertices in three different colors.



4) In any adjacent triangle, two of the vertices are already colored. Complete the coloring of the adjacent triangles using the unused color.

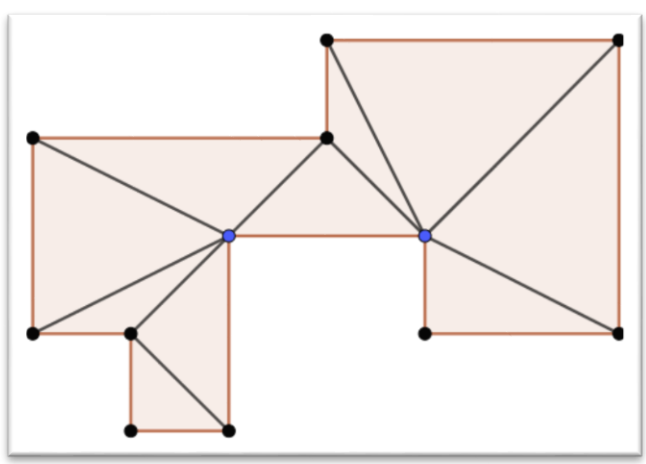


5) Continue the process until a polychromatic three coloring has been achieved (claim 2).



Fisk then states that because every triangle has a vertex of every color, each set of then every vertex set can see all of the triangles and thus have sight lines to the entire gallery. Therefore, any color set of vertices could represent guard positions, and the number of vertices in a particular set could represent the number of guards needed. Based on the triangulation, the blue and orange positions would require four guards and the yellow would require three. Every triangle can be colored in this way. If this is true then the color set of least magnitude represents the fewest guards given that triangulation. To establish an upper bound, one has to consider when the sets have the closest to an equal number of vertices. This occurs when the number of vertices (walls) is as close to $w/3$ as possible (divided by three as a result of the three coloring). Because all sets are integer values, at least one set, under most even circumstances, would have $\lfloor w/3 \rfloor$ guards. This establishes $g(w) \leq \lfloor w/3 \rfloor$ which, in turn, proves $g(w) = \lfloor w/3 \rfloor$.

It is important to note that this not only provides proof, but also provides an algorithm for determining locations of guards. It is also important to note, however, that this does not necessarily yield a least set of guards for a given gallery. In the example above it was determined that the yellow vertices would guard the gallery with three guards. However, as shown below, it could be accomplished with two.



Additional Related Theorems

Half Guards – Half Guards are guards that only have a 180 degree field of vision. An initial bound is projected as $2\lfloor w/3 \rfloor$ to represent two half guards back to back, but $\lfloor w/3 \rfloor$ guards suffice. This was proven by Csaba Tóth in 2000.

Half-guard theorem. Any art gallery with w walls can be guarded by $\lfloor w/3 \rfloor$ half-guards.

Right-Angled Galleries – Right-angled galleries have exclusively right angles. The proof used a convex quadrangulation approach, rather than triangulation, and a polychromatic four coloring. It was proven in 1985 by Jeffry Kahn, Maria Klawe, and Daniel Klettman.

Right-angled art gallery theorem. $\lfloor w/4 \rfloor$ guards are sufficient and sometimes necessary to protect a right-angled art gallery with w walls.

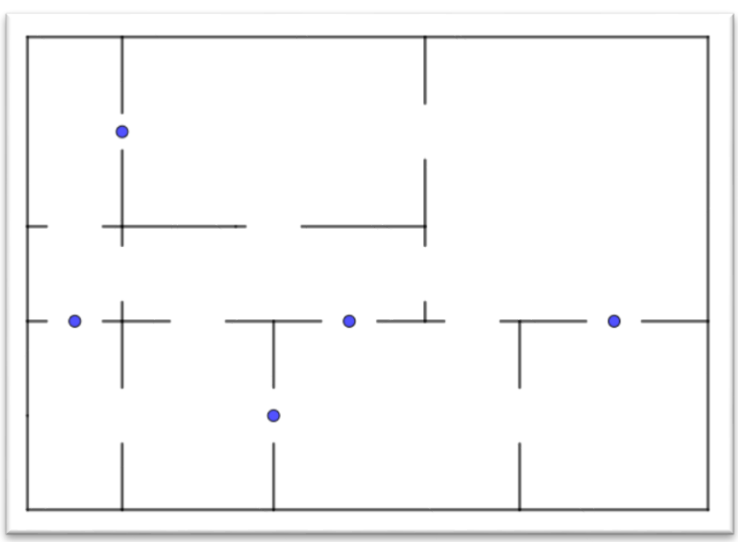
Guarded Guards – Guarded guards refer to when the gallery must be covered and every guard must be visible to at least one other guard. The proof follows Chvátal's argument.

Guarded guards theorem. $\left\lfloor \frac{3w-1}{7} \right\rfloor$ guards are sufficient, and sometimes necessary, to protect an art gallery with walls for $w = 5, 6, 7...$

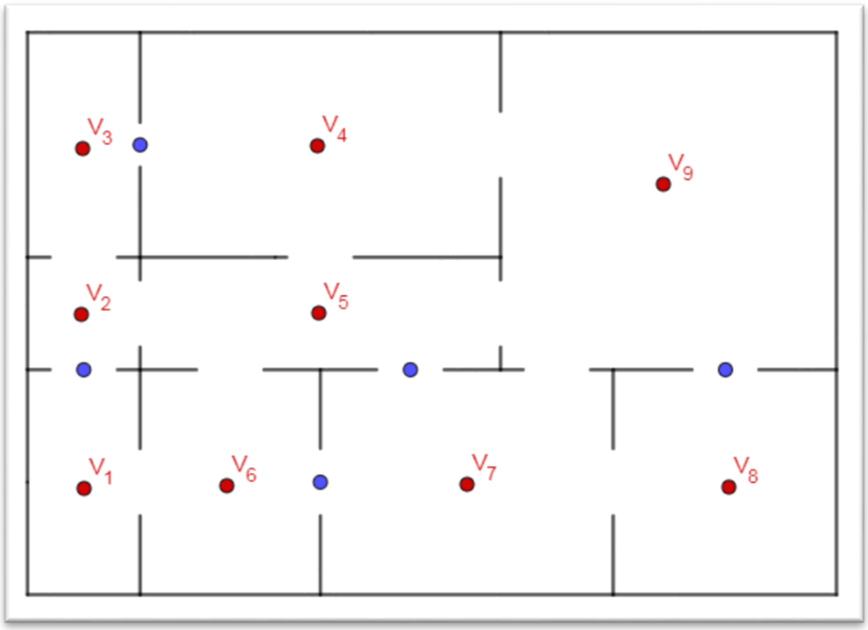
Guarded Guards for Right-Angled Galleries – This is a combination of the two previous scenarios and can be proven using an adaptation of Fisk's argument. First you complete a convex quadrangulation, then a triangulation so adjacent quadrilaterals don't triangulate using the same vertex.

Guarded guards theorem for right-angled galleries. $\lfloor w/3 \rfloor$ guards are sufficient, and sometimes necessary, to protect a right-angled art gallery with w walls for $w = 6, 8, 10...$

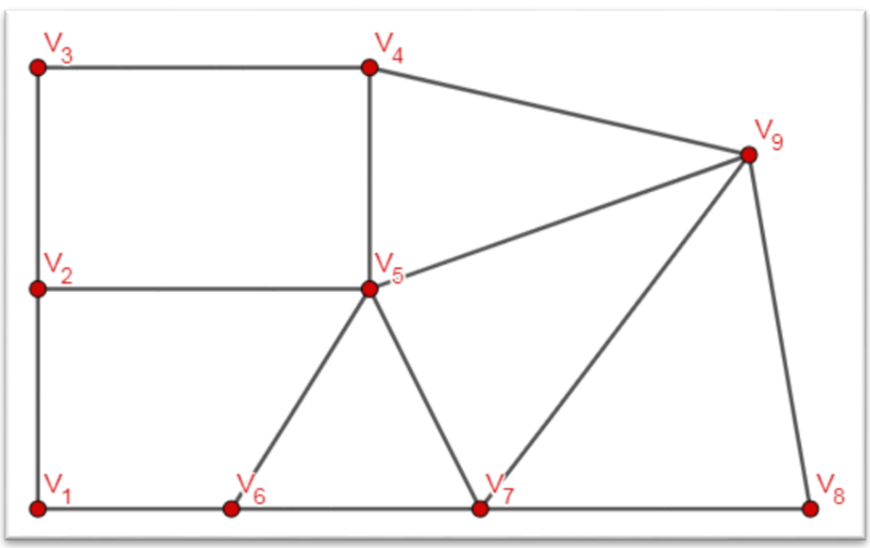
Rectangulated Galleries – These are galleries with multiple rectangular rooms. Every room has a single doorway to any room that shares a common wall section. In rectangulated galleries, guards are typically placed in doorways.



The proof of the Rectangulated gallery theorem is quite complicated, but has roots that relate to graph theory. It involves the use of the dual graph of the gallery created by placing vertices in each room



and connecting vertices if there is a door from one room to the other.



From this point, the proof becomes complicated but it based on the dual graph.

Rectangulated gallery theorem. Any rectangulated gallery with r rooms can be protected by $\lceil r/2 \rceil$ guards, but no fewer.

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